UNIT 13 PROBABILITY AND PROBABILITY RULES

STRUCTURE

13.0 OBJECTIVES

After studying this unit, you should be able to:

- comprehend the concept of probability,
- acquaint yourself with the terminology related to probability,
- understand the probability rules and their application in determining probability,
- differentiate between determination of probability under the condition of statistical independence and statistical dependence,
- apply probability concepts and rules to real life problems, and
- appreciate the relevance of the study of probability in decision making.

13.1 INTRODUCTION

In the previous units we have discussed the application of descriptive statistics. The subject matter of probability and probability rules provide a foundation for Inferential Statistics. There are various business situations in which the decision makers are forced to apply the concepts of probability. Decision making in various situations is facilitated through formal and precise expressions for the uncertainties involved. For instance formal and precise expression of stockmarket prices and product quality uncertainties, may go a long way to help analyse, and facilitate decision on portfolio and sales planning respectively. Probability theory provides us with the means to arrive at precise expressions for taking care of uncertainties involved in different situations.

This unit starts with the meaning of probability and its brief historical evolution. Its meaning has been described. The next section covers fundamental concept of probability as well as three approaches for determining probability. These approaches are : i) Classical approach; ii) Relative frequency of occurrence approach, and iii) Subjective approach.

Thereafter the addition rule for probability has been explained for both mutually exclusive events and non-mutually exclusive events. Proceeding further the unit addresses the important aspects of probability rules, the conditions of statistical independence and statistical dependence. The concept of marginal, joint, and conditional probabilities have been explained with suitable examples.

13.2 MEANING AND HISTORY OF PROBABILITY

Generally, in day-to-day conversation the words, probability, possible, chance, likelihood etc., are commonly used. You may have a rough idea of what is meant by these words. For example, we may come across the statements like: the train may come late today, the chance of winning the cricket match etc. It means there is uncertainity about the happening of the event(s). We live in a world where we are unable to forecast the future with complete certainty. Our need to cope with uncertainty leads us to the study and use of probability. In statistics, the term probability is established by definition and is not related to beliefs.

The concept of probability is as old as civilisation itself. As you know gambling is an age-old malaise. Gamblers have used the probability concept to make bets. The probability theory was first applied to gambling and later to other socio-economic problems. The probability theory was later on applied to the insurance industry, which evolved in the 19th century. This concept was used to determine the premium to be charged on the basis of probabilistic estimates of the life expectancy of the insurance policy holder. Consequently, the study of probability was initiated at many learning centers for students to be equipped with a tool for better understanding of many socio-economic phenomenon. Lately, the quantitative analysis has become the backbone of statistical application in business decision making and research.

If the conditions of certainty only were to prevail, life would have been much more simple. As is obvious there are numerous real life situations in which conditions of uncertainty and risk prevail. Consequently, we have to rely on the theory of chance or probability in order to have a better idea about the possible outcomes. There are social, economic and business sectors in which decision making becomes a real challenge for the managers. They may be in the dark about the possible consequences of their decisions and actions. Due to increasing competitiveness the stakes have become higher and cost of making a wrong decision has become enormous.

13.3 TERMINOLOGY

Before we proceed to discuss the fundamental concepts and approaches to determining probability, let us now acquaint ourselves with the terminology relevant to probability.

i) **Random Experiment:** A set of activities performed in a homogenous condition repetitively constitutes a random experiment. It results in various possible outcomes. An experiment, therefore, may be a single-trial, two-trial, or n-trial experiment. It may, thus, be noted that an experiment is determined in terms of the nature of trial and the number of times the trial is repeated.

ii) **Trial and Events:** To conduct an experiment once is termed as trial, while possible outcomes or combination of outcomes is termed as events. For example, toss of a coin is a trial, and the occurence of either head or a tail is an event.

iii) **Sample Space:** The set of all possible outcomes of an experiment is called the sample space for that experiment. For example, in a single throw of a dice, the sample space is (1, 2, 3, 4, 5, 6).

iv) **Collectively Exhaustive Events:** It is the set of all possible events that can result from an experiment. It is obvious that the sum total of probability value of each of these events will always be one. For example, in a single toss of a fair coin, the collectively exhaustive events are either head or tail. Since

$$P(H) = 0.5 \quad \text{and} \quad P(T) = 0.5$$

$$\therefore\ P(H) + P(T) = 0.5 + 0.5 = 1.0$$

v) **Mutually Exclusive Events:** Two events are said to be mutually exclusive events if the occurence of one event implies no possibility of occurence of the other event. For example, in throwing an unbiased dice, the occurence of the number at the top prevents the occurence of other numbers on it.

vi) **Equally Likely Events:** When all the possible outcomes of an experiment have an equal probability of occurance, such events are called equally likely events. For example, in case of throwing of a fair coin, we have already seen that

$$P(\text{Head}) = P\ (\text{Tail}) = 0.5$$

Many common experiments in real life also can have events, which have all of the above properties. The best example is that of a single toss of a coin, where both the possible outcomes or events of either head or tail coming on top are collectively exhaustive, mutually exclusive and equally likely events.

13.4 FUNDAMENTAL CONCEPTS AND APPROACHES TO PROBABILITY

Let us, now, discuss the concepts and approaches to determine and interpret probability. There are two fundamental concepts of probability. They are:

(i) The value of probability of any event lies between 0 to 1. This may be expressed as follows:

$$0 \le P\ (\text{Event}) \le 1$$

If the value of probability of an event is equal to zero, then the event is never expected to occur and if the probability value is equal to one, the event is always expected to occur.

(ii) The sum of the simple probabilities for all possible outcomes of an activity must be equal to one.

Before proceeding further, first of all, let us discuss different approaches to defining probability concept.

Approaches to Probability

There are three approaches to determine probability. These are :

a) **Classical Approach:** The classical approach to defining probability is based on the premise that all possible outcomes or elementary events of experiment are mutually exclusive and equally likely. The term equally likely means that each of all the possible outcomes has an equal chance of occurance. Hence, as per this approach, the probability of occuring of any event 'E' is given as:

$$P(E) = \frac{\text{No. of outcomes where the event occurs } [n(E)]}{\text{Total no. of all possible outcomes } (n(S)]}$$

This approach is also known as 'A Priori' probability as when we are performing the event using a fair coin, standard card, unbiased dice, then we can tell in advance the probability of happening of some event. We need not perform the actual experiment to find the required probability.

Example: When we toss a fair coin, the probability of getting a head would be:

$$P(\text{Head}) = \frac{\text{Total number of favourable outcomes}}{\text{Total no. of all possible outcomes}} = \frac{1}{1+1} = \frac{1}{2}$$

Similarly, when a dice is thrown, the probability of getting an odd number is $\frac{3}{6}$ or $\frac{1}{2}$.

The premise that all outcomes are equally likely assumes that the outcomes are symmetrical. Symmetrical outcomes are possible when a coin or a die being tossed are fair. This requirement restricts the application of probability only to such experiments which give rise to symmetrical outcomes. The classical approach, therefore, provides no answer to problems involving asymmetrical outcomes. And we do come across such situations more often in real life.

Thus, the classical approach to probability suffers from the following limitations: i) The approach cannot be useful in such a case, when it is not possible in the events to be considered "equally likely". ii) It fails to deal with questions like, what is the probability that a bulb will burn out within 2,000 hours? What is the probability that a female will die before the age of 50 years? etc.

b) **Relative Frequency of Occurrence:** The classical approach offers no answer to probability in situations where outcomes of an experiment lack of syummetry. Experiments of throwing a die which is loaded, or of tossing a coin which is biased, fall beyond the purview of classial approach since these experiments do not generate equally likely or symmetrical outcomes. In such cases the probability remains undefined. Failure of the classical approach on this count has made way for the relative frequency approach. It was in early 1800s, when several British statisticians started defining probability from statistical data collected on births and deaths. This concept was then widely used for calculating risk of loss of life as well as commercial insurance. According to this approach, the probability is the observed relative frequency of an event in a very large number of trials, when the conditions are stable.This method is utilised by taking relative frequencies of past occurrences as their probabilities.

Example: Supose an insurance company knows from available actuarial data, that for all males 50 years old, 30 out of 10,000 die within a one-year period.

From this available data the company can estimate the probability of death for that age group as 'P' where,

$$P = \frac{30}{10,000} = 0.003$$

Similarly, if the availability of fair complexioned girls is 1 in 20, then the

probability for that group is : $P = \frac{1}{20} = 0.05$.

This approach too has limited practical utility because the computation of probability requires repetition of an experiment a large number of times. This is practically true where an event occurs only once so that repetitive occurrences under precisely the same conditions is neither possible nor desirable.

c) **Subjective Probability:** The subjective approach to define probability was introduced by Frank Ramehs in 1926 in his book "The Foundation of Mathematics and Other Logical Essays". The subjective approach makes up for determining the probability of events where experiments cannot be performed repeatedly under the same conditions. According to this approach probability is based on the experience and the judgement of the person making this estimate. This may differ from person to person, depending on one's perception of the situation and the past experience. Subjective probability can be defined as based on the available evidence. Sometimes logic and past data are not so useful in determining the probability value, in those cases the subjective approach of the assesser is being used to find that probability. This approach is so flexible that it may be applied in a number of situations where the earlier two approaches fail to offer a satisfactory answer.

   Example: Suppose a commitee of experts constituted by the Government of India is faced with the decision, whether to allow construction of a nuclear power plant on a site which is close to a densely populated area. They will have to assign subjective probability to answer the question. "What is the probability of a nuclear accident at the site?" Obviously, neither of the previously discussed approaches can be applied in this case.

Impartantly, these three approaches compliment one another because where one fails, the other takes over. However, all are identical in as much as probability is defined as a ratio or a weight assigned to the likelylihood of occurrence of an event.

Self Assessment Exercise A

1) What does it mean when:

   i) the probability of an event is zero.

   ii) the probability of an event is one.

   iii) all the possible outcomes of an experiment have an equal chance of occurance.

   iv) the two events not occurring simultaneously, have some elements common to both.

2) You are being asked to draw one card from a deck of 52 cards. In these 52 cards, 4 categories exist namely: spade, club, diamond, and heart (i.e. 13 each). So you can test your understanding of the concept by going through the

following cases by writing down yes/no in the second and third column given below:

	Draws	Mutually Exclusive	Collectively Exclusive
a)	Draw a spade and a club		
b)	Draw an ace and a three		
c)	Draw a club and a non-club		
d)	Draw a 5 and a diamond		

13.5 PROBABILITY RULES

The related terms and concepts defining probability, which we have discussed above, are needed to develop probability rules for different types of events.

The following rules of probability are useful for calculating the probability of an event/events under different situations.

13.5.1 Addition Rule for Mutually Exclusive Events

If two events, A and B, are mutually exclusive, then the probability of occurance of either A or B is given by the following formula:

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

If A and B are mutually exclusive events, this rule is depicted in Figure 13.1, below.

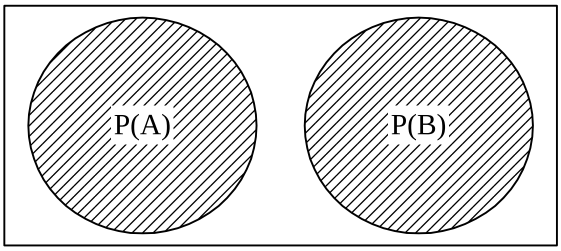


Figure: 13.1

The essential requirement for any two events to be mutually exclusive is that there are no outcomes common to the occurance of both. This condition is satisfied when sample space does not contain any outcome favourable to the occurance of both A and B means $A \cap B = \phi$

Let us consider the following illustration to understand the concept.

Illustration 1: In a game of cards, where a pack contains 52 cards, 4 categories exist namely spade, club, diamond, and heart. If you are asked to draw a card from this pack, what is the probability that the card drawn belongs to either spade or club category.

Solution: Here, P (Spade or club) = P (Spades) + P (Club)

Where $P(\text{Spade}) = \frac{13}{52} = \frac{1}{4}$ and $P(\text{Club}) = \frac{13}{52} = \frac{1}{4}$

$\therefore P(\text{Spade or Club}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

There is an important special case for any event E, either E happens or it does not. So, the events E and not E are exhaustive and exclusive.

So, P (E) + P (not E) = 1

or, P (E) = 1–P (not E)

Sometimes P (not E) is also written as either $P(E^{|}) = 1 - P(\overline{E})$

So, $P\,(E) = 1 - P\,(E^{|}) = 1 - P(\overline{E})$.

13.5.2 Addition Rule for Non-Mutually Exclusive Events

Non-mutually exclusive (overlapping) events present another significant variant of the additive rule. Two events (A and B) are not mutually exclusive if they have some outcomes common to the occurance of both, then the above rule has to be modified in order to account for the overlapping areas, as it is clear from Figure 13.2. below.

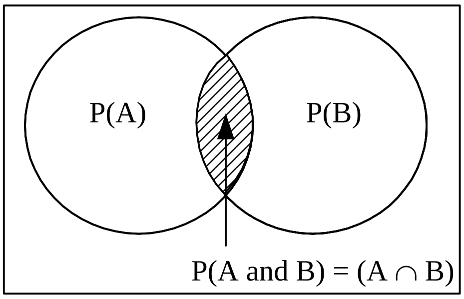


Figure: 13.2

In this situation, the probability of occurrence of event A or event B is given by the formula

$$P\ (A \text{ or } B) = P\ (A \cup B) = P\ (A) + P\ (B) - P\ (A \text{ and } B)$$

where P (A and B) is the joint probability of events A and B, i.e., both occuring together and is usually written as P $(A \cap B)$.

Thus, it is clear that the probability of outcomes that are common to both the events is to be subtracted from the sum of their simple probability.

Consider the following illustrations to understand the application of this concept.

Illustration 2: The event of drawing either a Jack or a spade from a well-shuffled deck of playing cards. Find the probability.

Solution: These events are not mutually exclusive, so the required probability of drawing a Jack or a spade is given by:

P (Jack or Spade) = P (Jack) + P (Spade) – P (Jack and Spade)

$$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

Illustration 3: The employees of a certain company have to elect one out of five of their numbers for the employee–management relationship committee. Details of these five were as follows:

Sex	Age
Male	40
Female	20
Male	32
Female	45
Male	30

Find the probability of selecting a representative that would be either male or over 35.

Solution: P (Male or over 35) = P (Male) + P (over 35) – P (Male and over 35)

$$= \frac{3}{5} + \frac{2}{5} - \frac{1}{5} = \frac{4}{5}$$

Self Assessment Exercise B

1) An urn contains 75 marbles: 35 are blue, and 25 of these blue marbles are swirled. The rest of them are red, and 30 of the red ones are swirled. The marbles that are not swirled are clear. What is the probability of drawing:

   a) A blue marble from the urn?

   b) A clear marble from the urn?

   c) A blue, swirled marble?

   d) A red, clear marble?

   e) A swirled marble?

   ....

   ....

   ....

   ....

   ....

   ....

Proceeding further with the multiplication rule, it is pertinent to discuss the concept of statistical independency and statistical dependency of events.

13.6 PROBABILITY UNDER STATISTICAL INDEPENDENCE

The two or more events are termed as statistically independent events, if the occurrence of any one event does not have any effect on the occurrence of any other event. For example, if a fair coin is tossed once and suppose head comes, then this event has no effect in any way on the outcome of second toss of that same coin. Similarly, the results obtained by drawing hearts from a pack has no effect in any way on the results obtained by throwing a dice. These events thus are being termed as statistically independent events. There are three trypes of probability under statistically independent case.

a) Marginal Probability;

b) Joint Probability;

c) Conditional Probability.

Let us discuss each one of them, one by one.

a) **Marginal Probability Under Statistical Independence**

A Marginal/Simple/Unconditional probability is the probability of the occurrence of an event. For example, in a fair coin toss, probability of having a head is:

$$P(H) = ½ = 0.5$$

Therefore, the marginal probability of an event (i.e. having a head) is 0.5. Since, the subsequent tosses are independent of each other, therefore, it is a case of statistical independence.

Another example can be given in a throw of a fair die, the marginal probability of the face bearing number 3, is:

$$P(3) = 1/6 = 0.166$$

Since, the tosses of the die are independent of each other, this is a case of statistical independence.

b) **Joint Probability Under Statistical Independence**

This is also termed as "**Multiplication Rule of Probability**". In many situations we are interested in finding out the probability of two or more events either occurring together or in quick succession to each other, for this purpose the concept of joint probability is used.

This joint probability of two or more statistically independent events occurring together is determined by the product of their marginal probability. The corresponding formula may be expressed as:

$$P(A \text{ and } B) = P(A) \times P(B)$$

Similarly, it can be extended to more than two events also as:

$$P(A \text{ and } B \text{ and } C) = P(A) \times P(B) \times P(C) \text{ and so on.}$$

i.e. $P(A \text{ and } B \text{ and } C \text{ and } \ldots) = P(A) \times P(B) \times P(C) \times \ldots$

For instance, when a fair coin is tossed twice in quick succession, the probability of head occuring in both the tosses is:

$$P(H_1 \text{ and } H_2) = P(H_1) \times P(H_2)$$
$$= 0.5 \times 0.5 = 0.25$$

Where, H_1 is the occurrence of head in 1st toss, and H_2 is the occurrence of head in 2nd toss.

Take another example: When a fair die is thrown twice in quick succession, then to find the probability of having 2 in the 1st throw and 4 in second throw is, given as:

P (2 in 1st throw and 4 in 2nd throw)

= P (2 in the 1st throw) × P (4 in the 2nd throw)

$$= \frac{1}{6} \times \frac{1}{6} = \frac{1}{36} = 0.028$$

c) Conditional Probability Under the Condition of Statistical Independence

The third type of probability under the condition of statistical independence is the Conditional Probability. It is symbolically written as P (A/B), i.e., the conditional probability of occurrence of event A, on the condition that event B has already occurred.

In case of statistical independence, the conditional probability of any event is akin to its marginal probability, when both the events are independent of each other.

Therefore, P (A/B) = P (A), and
P (B/A) = P (B).

For example, if we want to find out what is the probaility of heads coming up in the second toss of a fair coin, given that the first toss has already resulted in head. Symbolically, we can write it as:

$P\ (H_2/H_1)$

As, the two tosses are statistically independent of each other

so, $P\ (H_2/H_1) = P\ (H_2)$

The following table 13.1 summarizes these three types of probabilities, their symbols and their mathematical formulae under statistical independence.

Table 13.1

Probability's Type	Symbol	Formula
Marginal	P (A)	P (A)
Joint	P (AB)	P (A) × P (B)
Conditional	P (B/A)	P (B)

For example, if the first child of a couple is a girl to find the probability that the second child will be a boy. In this case:

P (B/G) = P (B)

As both the events are independent of each other, the conditional probability of having the second child as a boy on condition that their first child was a girl, is equal to the marginal probability of having the second child as a boy.

For example, take the case of rain in different states of India. Suppose, the probability of having rain in different states of India is given as:

P (rainfall in Bihar) = 0.8

P (rainfall in Uttar Pradesh) = 0.75

P (rainfall in Rajasthan) = 0.4

P (rainfall in Gujarat) = 0.5

Then, to find out probability of having rainfall in Gujarat, on condition that during this period, Bihar receives a heavy rainfall, is a case of statistical independence, as both the events (rain in Gujarat and rain in Bihar) are quite independent to each other. So, this conditional probability is equal to marginal probability of having rainfall in Gujarat (which is equal to 0.5).

For example, an urn contains 3 white balls and 7 black balls. We draw a ball from the Urn, replace it and then again draw a second ball. Now, we have to find the probability of drawing a black ball in the second draw on condition that the ball drawn in the first attempt was a white one.

This is a case of conditional probability, but is equal to the marginal (simple) probability, as the two drawn are independent events.

$$\therefore P\ (B/W) = P\ (B) = 7/10 = 0.7$$

Self Assessment Exercise C

1) Which of the following pair of events are statistically independent ?

   a) The time until the failure of a watch and of a second watch marketed by different companies – yes/no

   b) The life span of the current Indian PM and that of current Pakistani President – Yes/no.

   c) The takeover of a company and a rise in the price of its stock – Yes/no.

2) What is the probability that a couple's second child will be

   a) A boy, given that their first child was a girl?

   b) A girl, given that their first child was a girl?

   ....

3) What is the probability that in selecting two cards one at a time from a deck with replacement, the second card is

   a) A face card, given that the first card was red?

   b) An ace, given that the first card was a face card?

   c) A black jack, given that the first card was a red ace?

   ....

   ....

   ....

4) A bag contains 32 marbles: 4 are red, 9 are black, 12 are blue, 6 are yellow and 1 is purple. Marbles are drawn one at a time with replacement. What is the probability that:

   a) The second marble is yellow given the first one was yellow?

   b) The second marble is yellow given the first one was black?

   c) The third marble is purple given both the first and second were purple?

....

....

....

13.7 PROBABILITY UNDER STATISTICAL DEPENDENCE

Two or more events are said to be statistically dependent, if the occurrence of any one event affects the probability of occurrence of the other event.

There are three types of probability under statistical dependence case. They are:

a) Conditional Probability;

b) Joint Probability;

c) Marginal Probability

Let us discuss the concept of the three types.

a) Conditional Probability Under Condition of Statistical Dependence

We shall first discuss the concept of conditional probability, as it forms the basis for joint and marginal probabilities concept under statistical dependence.

The conditional probability of event A, given that the event B has already occured, can be calculated as follows:

$$P\,(A\,/\,B) = \frac{P\,(AB)}{P\,(B)}$$

where, P (AB) is the joint probability of events A and B.

This is also referred to as **Bayes' Theorem.**

Let us consider the following illustration to understand this concept.

Illustration 4: (i) A box containing 10 balls which have the following distribution on the basis of colour and pattern.

- 3 are coloured and dotted.
- 1 is coloured and sptripped.
- 2 are grey and dotted.
- 4 are grey and stripped.

a) Suppose someone draws a coloured ball from the box. Find what is the probability that it is (i) dotted and (ii) it is stripped ?

Solution: The problem can be expressed as P (D/C) i.e., the conditional probability that the ball drawn is dotted given that it is coloured.

Now from the information given in the question.

(i) P(CD) = 3/10 = Joint Probability of drawn ball becoming a coloured as well as a dotted one.

Similarly, P (CS) = 1/10, P (GD) = 2/10, and P (GS) = 4/10

$$\text{So,} = P(D/C) = \frac{P(DC)}{P(C)}$$

where, P(C) = Probability of drawing a coloured ball from the box = 4/10 (4 coloured balls out of 10 balls).

$$\therefore P\,(D/C) = \frac{3/10}{4/10} = 0.75$$

ii) Similarly, P(S/C) = Conditional probability of drawing a stripped ball on the condition of knowing that it is a coloured one.

$$= \frac{P(SC)}{P(C)} = \frac{1/10}{4/10} = 0.25$$

Thus, the probability of coloured and dotted ball is 0.75. Similarly, the probability of coloured and stripped ball is 0.25.

b) Continuing the same illustration, if we wish to find the probability of (i) P (D/G) and (ii) P (S/G)

Solution: i) $P\,(D/G) = \frac{P\,(DG)}{P\,(G)} = \frac{2/10}{6/10} = \frac{1}{3} = 0.33$

where, P(G) = Total probability of grey balls, i.e., 6/10 and

ii) $(S/G) = \frac{P\,(SG)}{P\,(G)} = \frac{4/10}{6/10} = \frac{2}{3} = 0.66$

c) Similarly, to calculate (i) P (G/D) and (ii) P (C/D)

Solution: (i) $P\,(G/D) = \frac{P\,(GD)}{P\,(D)} = \frac{2/10}{5/10} = 0.4$

and (ii) $P\,(C/D) = \frac{P\,(CD)}{P\,(D)} = \frac{3/10}{5/10} = 0.6$

d) If we wish to find (i) P (C/S) and (ii) P (G/S),

Solution: (i) $P\,(C/S) = \frac{P\,(CS)}{P\,(S)} = \frac{1/10}{5/10} = 0.2$

(ii) $P\,(G/S) = \frac{P\,(GS)}{P\,(S)} = \frac{4/10}{5/10} = 0.8$

b) Joint Probability Under the Condition of Statistical Dependence

This is an extension of the **multiplication rule of probability** involving two or more events, which have been discussed in the previous section 13.6, for calculating joint probability of two or more events under the statistical independence condition.

The formula for calculating joint probability of two events under the condition of statistical independence is derived from the formula of **Bayes' Theorem**.

Therefore, the joint probability of two statistically dependent events A and B is given by the following formula:

$$P\ (AB) = P\ (A/B) \times P\ (B)$$

$$\text{or } P\ (BA) = P\ (B/A) \times P\ (A)$$

depending upon whether order of occurrence of two events is B, A or A, B.

Since, P (A/B) = P (B/A), So the product on the RHS of the formula must also be equal to each other.

$$\therefore\ P\ (A/B) \times P\ (B) = P\ (B/A) \times P\ (A)$$

Notice that this formula is not the same under conditions of statistical independence, i.e., P (BA) = P(B) × P (A). Continuing with our previous illustration 4, of a box containing 10 balls, the value of different joint probabilities can be calculated as follows:

Converting the above general formula i.e., P (AB) = P (A/B) × P (B) into our illustration and to the terms coloured, dotted, stripped, and grey, we would have calculated the joint probabilities of P (CD), P (GS), P (GD), and P (CS) as follows:

i) P (CD) = P (C/D) × P (D) = 0.6 × 0.5 = 0.3

ii) P (GS) = P (G/S) × P (S) = 0.8 × 0.5 = 0.4

iii) P (GD) = P (G/D) × P (D) = 0.4 × 0.5 = 0.2

iv) P (CS) = P (C/S) × P (S) = 0.2 × 0.5 = 0.1

Note: The values of P (C/D), P (G/S), P (G/D), and P (C/S) have been already computed in conditional probability under statistical dependence.

c) Marginal Probability Under the Condition of Statistical Dependence

Finally, we discuss the concept of marginal probability under the condition of statistical dependence. It can be computed by summing up all the probabilities of those joint events in which that event occurs whose marginal probability we want to calculate.

Illustration 5: Consider the previous illustration 4, to compute the marginal probability under statistical dependence of the event: i) dotted balls occured, ii) coloured balls occured, iii) grey balls occured, and iv) stripped balls occured.

Solution: i) We can obtain the marginal probability of the event dotted balls by adding the probabilities of all the joint events in which dotted balls occured.

$$P\ (D) = P\ (CD) + P\ (GD) = 3/10 + 2/10 = 0.5$$

In the same manner, we can compute the joint probabilities of the remaining events as follows:

ii) P (C) = P (CD) + P(CS) = 3/10 + 1/10 = 0.4

iii) P (G) = P (GD) + P (GS) = 2/10 + 4/10 = 0.6

iv) P (S) = P (CS) + P (GS) = 1/10 + 4/10 = 0.5

The following table 13.2 summarizes three types of probabilities, their symbols and their mathematical formulae under statistical dependence.

Table 13.2: Probabilities under Statistical Dependence

Probability Type	**Symbol**	**Formula**
Marginal	P (A)	Sum of the probabilities of joint events in which 'A' occurs
Joint	P (AB) or P (BA)	P (A/B) × P (B) P (B/A) × P (A)
Conditional	P (A/B) or P (B/A)	P (A/B) / P (B) P (B/A) / P (A)

Self Assessment Exercise D

1) According to a survey, the probability that a family owns two cars of annual income greater than Rs. 35,000 is 0.75. Of the households surveyed, 60 per cent had incomes over Rs. 35,000 and 52 per cent had two cars. What is the probability that a family has two cars and an income over Rs. 35,000 a year?

....

....

....

....

2) Given that P(A) = 3/14, P (B) = 1/6, P(C) = 1/3, P (AC) = 1/7 and P (B/C) = 5/21, find the following probabilities: P (A/C), P (C/A), P (BC), P (C/B).

....

....

....

....

3) At a restaurant, a social worker gathers the following data. Of those visiting the restaurant, 59% are male, 32 per cent are alcoholics and 21 per cent are male alcoholics. What is the probability that a random male visitor to the restaurant is an alcoholic?

....

....

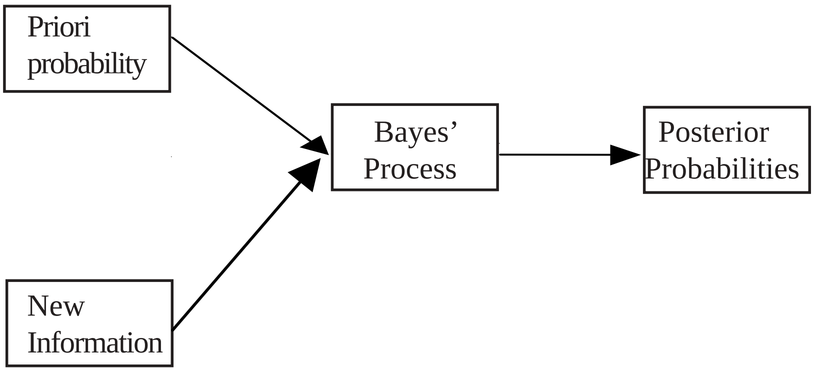
....

13.8 BAYES' THEOREM: REVISION OF A-PRIORI PROBABILITY

As discussed earlier, the basic objective of calculating probabilities is to facilitate us in decision-making. For example, assume that you are a seller of winter garments. Obviously, you are interested in the demand of winter garments. To help you in deciding on the amount you should stock for this winter, you have computed the probability of selling different quantities and have noted that the chance of selling a certain quantity is very high. Accordingly, you have taken the decision to stock a large quantity of the product. Suppose, when, finally the season ends, you find that you are left with a large quantity of stock. Then you feel that the earlier probability calculation should be revised given the new experience to help you to decide on the stock for the next winter. Similarly situations exist where we are interested in an event on an on-going basis. Every time some new information is available, we revise our probability. This revision of probability with added information is formalised in the probability theory in terms of a theorem called Bayes' theorem. Bayes' theorem offers a powerful statistical method of combining our evaluation of new information as well as our prior estimate of the probability to create Posterior Probability.

Thus, probabilities before revision by Bayes' concept are termed as **Priori probabilities**. Probabilities which have undergone revision in the light of additional information by Bayes' rule are termed as **Posterior probabilities** or revised probabilities which can be altered after additional information is gathered.

Bayes' theorem can be illustrated by the following figure.



(Source: Quantitative Analysis for Management; Render & Stair)

The origin of the concept of computing posterior probabilities with additional information is attributable to the Reverend Thomas Bayes (1702-1761). Bayes' theorem is based on the formula for conditional probability under statistical dependence, discussed in section 13.7, i.e.,

$$P(A/B) = \frac{P(A/B)}{P(B)}$$

It is worthwhile to note that revised probabilities are, thus, always conditional probabilities.

Let us consider the following illustration to understand the application of Bayes' theorem.

Illustration 6: Suppose a box cantains a fair (unbiased) and a loaded (biased) dice. Naturally, the probability of having 3 on the roll of the fair die is 1/6 = 0.166. However, the same probability on the loaded die is suppose 0.60. Suppose, we do not have an idea of which one is loaded and which one is fair. We select one die and roll it. The result comes out as 3. Now, we have this new information which can be used it to find out the probability that the die rolled was a (i) fair (ii) loaded.

Here, we have

P (L) = 0.5; P (F) = 0.5; P (3/F) = 0.166; and P (3/L) = 0.600.

Here, we are going to calculate the joint probabilities of P (3 and F) and P (3 and L), using the formula:

P (AB) = P (A/B) × P (B)

So, P (3 and F) = P (3/F) × P (F) = 0.166 × 0.5 = 0.083 and

P (3 and L) = P (3/L) × P (L) = 0.600 × 0.5 = 0.30

Further, we have to calculate the value of P (3), which is:

P (3) = P (3F) + P (3L)

= 0.083 + 0.300 = 0.383

Now, we can find out the value of P (F/3), as well as P (L/3), by using the formula

$$P(F/3) = \frac{P(F \text{ and } 3)}{P(3)} = \frac{0.083}{0.383} = 0.216 \text{ and}$$

$$P(L/3) = \frac{P(L \text{ and } 3)}{P(3)} = \frac{0.300}{0.383} = 0.784$$

These two conditional probabilities are called the **Revised/Posterior Probability**.

Our original estimates of probability of fair die being rolled was 0.5 and similarly for loaded die was again 0.5. But, with the single roll of the die, the probability of the loaded die being rolled is given that 3 has appeared on the top, increases to 0.78, while for rolled die to be the fair one decreases to 0.22. This example illustrated the power of Bayes’s theorem.

Self Assessment Exercise E

1. There are two machines, A and B, in a factory. As per the past information, these two machines produced 30% and 70% of items of output respectively. Further, 5% of the items produced by machine A and 1% produced by machine B were defective. If a defective item is drawn at random, what is the probability that the defective item was produced by machine A or machine B?

....

....

....

....

....

....

....

....

....

....

13.9 LET US SUM UP

At the beginning of this unit the historical evolution and the meaning of probability has been discussed. Contribution of leading mathematicians has been highlighted. Fundamental concepts and approaches to determining probability have been explained. The three approaches namely; the classical, the relative frequency, and the subjective approaches are used to determine the probability in case of risky and uncertain situation have been discussed.

Probability rules for calculating probabilities of different types of events have been explained. Further the condition of statistical independence and statistical dependence have been defined. Three types of probabilities namely: marginal, joint and conditional under statistical independence and statistical dependence have been explained. Finally, the Bayesian approach to the revision of a priori probability in the light of additional information has been undertaken.

13.10 KEY WORDS

Bayes' Theorem: It gives us a formula that computes the conditional probabilities when dealing with statistically dependent events.

Classical/Logical Approach: An objective way of assessing probabilistic value based on logic.

Collectively Exclusive Event: This is the collection of all possible outcomes of an experiment.

Conditional Probability: The probability of the happening of an event on the condition that another event has already occurred.

Dependent Event: This is the situation in which the occurrence of one event affects the happening of another event.

Independent Event: This is the situation in which the occurrence of an event has no effect on the probability of the occurrence of any other event.

Joint Probability: The probability of occuring of events together or in quick succession

Marginal/Simple Probability: As the name suggests, it is the simple probability of occurrence of an event.

Mutually Exclusive Events: A situation in which only one event can occur on any given trial/experiment. It means events that cannot occur together.

Posterior or Revised Probability: A probability value that results from new or revised information of priori probabilities.

Priori Probability: A probability value determined before new/additional information is obtained.

Probability: Any numerical value between 0 and 1, both inclusive, telling about the likelihood of occurrence of an event.

Relative Frequency Approach: An objective way of determining probability based on observing frequency over a number of trials.

Subjective Approach: A way of determining probability values based on experience/judgement.

13.11 ANSWERS TO SELF ASSESSMENT EXERCISES

A) 1. i) It is an impossible event

ii) It is an event which must occur

iii) They are equally likely events

iv) They are mutually exclusive events

2. a) No–Yes; b) No–Yes; c) No–No; d) Yes–No

B) 1. a) 7/15; b) 4/15; c) 11/15; d) 2/3; e) 11/15.

C) 1. a) Yes; b) Yes; c) No.

2. a) 1/2; b) 1/2.

3. a) P $(Face_2/Red_1) = 3/13$

b) P $(Ace_2/Face_1) = 1/13$

c) P (Black $Jack_2$/Red Ace_1) = 1/26.

4. a) 6/32; b) 6/32; c) 1/32.

D) 1. Let I = income > Rs. 35,000; C = 2 cars
P (C and I) = P (C/I) P (I) = 0.75 (0.6) = 0.45.

2. 3/7; 2/3; 5/63; 10/21.

3. 0.356.

E) 1. Machine A = 0.682; Machine B = 0.318.

Supplementary Illustrations

1) If A and B are two non-mutually exclusive events, such that probability of happening of event A is 0.25, probability of the happening of event B is 0.4 and probability of happening of event A or B is 0.5. Then we have to find out the value of probability of happening of both the events together.

Here, we have

$P(A) = 0.25$

$P(B) = 0.40$ and

$P(A \cup B) = 0.5$, then $P(A \cap B) = ?$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Replacing given values in this equation, we have

$0.5 = 0.25 + 0.40 - P(A \cap B)$

or $P(A \cap B) = 0.15$

2) Find out the probability of at least one tail on three tosses of a fair coin.

As there is only one case, where there is no tail on three coins. This case is of having H_1, H_2, H_3.

Now

$P(H_{\backslash 1}\ H_2\ H_3) = P(H_1) \times P(H_2) \times P(H_3)$

$= 0.5 \times 0.5 \times 0.5 = 0.125$

$\therefore$ To get the answer, we just need to subtract this probability from 1. So, that our answer is:

$= 1–P(H_1\ H_2\ H_3) = 1–0.125 = 0.875$

Here, the probability of at least one tail occurring in three consecutive tosses is 0.875.

3) What is the chance that a non-leap year contain 53 Mondays?

A non-leap year consists of 365 days, i.e., a total 52 full weeks and one extra day. So, a non-leap year contain 53 Mondays, only when that extra day must also be a Monday.

But, as that day can be from any of the following set, viz., either Sunday or Monday or Wednesday or Thursday or Friday or Saturday.

So, the required probability $\dfrac{\text{No. of favourable events}}{\text{Total no. of outcomes}} = \dfrac{1}{7}$

4) What is the probability of having at least one head on two tosses of a fair coin?

The possible ways in which a head may occur are $H_1\ H_2$; $H_1\ T_2$; $T_1\ H_2$. Each of these has a probability of 0.25.

As $P(H_1\ H_2) = P(H_1) \times P(H_2)$

$= 0.5 \times 0.5 = 0.25.$

The results are similar for $P(H_1\ T_2)$ and $P(T_1\ H_2)$ also. Since, the two tosses are statistically independent events. Therefore, the probability of at least one head on two tosses is $0.25 \times 3 = 0.75$.

5) Suppose we are tossing an unfair coin, where the probability of getting head in a toss is 0.8. If we have to calculate the probability of having three heads in three consecutive trials

Then, as given

$P(H) = 0.8, \therefore P(T) = 0.2$

As given if $P(H_1\ H_2\ H_3)$ represents the probability of having three heads in three trials.

And $P(H_1\ H_2\ H_3) = P(H_1) \times P(H_2) \times P(H_3)$

$= 0.8 \times 0.8 \times 0.8 = 0.512$

If we have to calculate the probability of having three consecutive tails in three trials.

In that case $P(T_1 T_2 T_3) = P(T_1) \times P(T_2) \times P(T_3)$

$= 0.2 \times 0.2 \times 0.2 = 0.008$

6) Let us consider an urn having 10 balls with descriptions given below:

4 are White (W) and lettered (L)

2 are White (W) and numbered (N)

3 are Yellow (Y) and lettered (L)

1 are Yellow (Y) and numbered (N).

Suppose at random one ball in picked out from the urn, then we have to find out the probability that:

i) The ball is lettered

ii) The ball drawn is lettered, given that it is yellow

For this, first of all, let us tabulate a series of useful probabilities.

P (WL) = 0.4

P (YL) = 0.3

P (WN) = 0.2

P (YN) = 0.1

Also, P (W) = 0.6

P (Y) = 0.4

P (L) = 0.7

P (N) = 0.3

As we also knew that:

P (W) = P (WL) + P (WN) = 0.4 + 0.2 = 0.6

P (Y) = P (YL) + P (YN) = 0.3 + 0.1 = 0.4

P (L) = P (WL) + P (YL) = 0.4 + 0.3 = 0.7

and P (N) = P (WN) + P (YN) = 0.2 + 0.1 = 0.3

So, (i) (L) = 0.7

(ii) for $P(L/Y) = \dfrac{P(LY)}{P(Y)} = \dfrac{0.3}{0.4} = 0.75$

and $P(C/D) = \dfrac{P(CG)}{P(D)} = \dfrac{3/10}{5/10} = 0.6$

7) A manufacturing firm produces units of a product in three machines. From the past records, the proportions of defectives produced at each machine the following conditional probabilities are computed.

$P(D/M_1) = 0.06$; $P(D/M_2) = 0.15$; $P(D/M_3) = 0.10$.

Events M_1, M_2, M_3 are unit produced in Machines 1, 2, and 3 respectively.

Event D is a defective unit.

The first machine produces 30% of the units of the product, the second machine 20% and the third machine 50%. A unit of the product produced at one of these machines is tested and found to be defective. What are the probabilities that the defective unit was produced in any of the three machines?

Solution: Here, we have

$P(M_1) = 0.3$, $P(H_2) = 0.2$, $P(M_3) = 0.5$ (Check

$P(M_1) + P(M_2) + P(M_3) = 1$, follows from definition mutually exclusive and collectively exchaustive events).

and also, $P(D/M_1) = 0.06$, $P(D/H_2) = 0.15$; $P(D/M_3) = 0.10$.

Now, we have to calculate the joint probabilities of P (D and M_1), P (D and M_2), and P (D and M_3), by using the formula:

$$P(AB) = P(A/B) \times P(B)$$

$$\therefore\ P(DM_1) = P(D/M_1) \times P(M_1) = 0.06 \times 0.3 = 0.018$$

$$P(DM_2) = P(D/M_2) \times P(M_2) = 0.15 \times 0.2 = 0.03$$

$$P(DM_3) = P(D/M_3) \times P(M_3) = 0.10 \times 0.5 = 0.05$$

Further we can obtain P (D) by adding the above obtained joint probabilities i.e.,

$$P(D) = P(DM_1) + P(DM_2) + P(DM_3)$$

$$= 0.018 + 0.03 + 0.05 = 0.098$$

Finally, we can find at the values of $P(M_1/D)$, $P(M_2/D)$ and $P(M_3/D)$

$$P(M_1/D) = \frac{P(M_1 \text{ and } D)}{P(D)} = \frac{0.018}{0.098} = 0.1837$$

$$P(M_2/D) = \frac{P(M_2 \text{ and } D)}{P(D)} = \frac{0.03}{0.098} = 0.3061$$

$$P(M_3/D) = \frac{P(M_3 \text{ and } D)}{P(D)} = \frac{0.05}{0.098} = 0.5102$$

$$\overline{\underline{\ 1.000\ }}$$

These three conditional probabilities are called the posterior probabilities.

It is clear from the revised probability values that the probabilities of defective units produced in M_1 is 0.18, M_2 is 0.31 and M_3 is 0.51 against the past probabilities 0.3, 0.2, and 0.5 respectively. And the probability that a defective unit produced by this firm is 0.098.

13.12 TERMINAL QUESTIONS/EXERCISES

1. Why do we study probability? Explain its importance and relevance.

2. Define the following, using appropriate examples:

   i) Equally likely events

   ii) Mutually exclusive events

   iii) Trial and event

   iv) Sample space

3. What are the different approaches to probability? Explain with suitable examples.

4. State and prove the addition rule of probability for two mutually exclusive events.

5. Explain the types of probability under statistical independence.

6. Explain the use of Bayes' theorem in probability.

7. One ticket is drawn at random from an urn containing tickets numbered from 1 to 50. Find out the probability that:

   i) It is a multiple of 5 or 7

   ii) It is a multiple of 4 or 3

   [Answer: i) 17/50, ii) 12/25]

8. If two dice are being rolled, then find out the probabilities that:

   i) The sum of the numbers shown on the dice is 7.

   ii) The numbers shown on the dice are equal.

   iii) The number shown by second die is greater than the number shown by the first die.

   [Answer: (i) 1/6 (ii) 1/6, (iii) 5/12]

9. (a) Find out the probability of getting head in a throw of a fair coin.

   (b) If two coins are tossed once, what is the probability of getting

   (i) Both heads

   (ii) At least one head ?

   [Answer: (a) 1/2 (b) (i) 1/4 (ii) 3/4]

10. Given that P(A) = 3/14P (B) = 1/6, P (C) = 1/3, P (AC) = 1/7, P (B/C) = 5/21.

    Find out the values of P (A/C), P (C/A), P (B/C), P (C/B).

    [Ans. 3/7, 2/3, 5/63, 10/21]

11. A T.V. manufacturing firm purchases a certain item from three suppliers X, Y and Z. They supply 60%, 30% and 10% respectively. It is known that 2%, 5% and 8% of the items supplied by the respective suppliers are defective. On a particular day, the firm received items from three suppliers and the contents get mixed. An item is chosen at random:

    a) What is the probability that it is defective?

    b) If the item is found to be defective what is the probability that it was supplied by X, Y, and Z?

    [Ans. P (D) = 0.035 P (X/D) = 0.34, P (Y/D) = 0.43, and P (Z/D) = 0.23].

Note: These questions/exercises will help you to understand the unit better. Try to write answers for them. But do not submit your answers to the university for assessment. These are for your practice only.

13.13 FURTHER READING

The following text books may be used for more indepth study on the topics dealt within this unit.

Levin, R.I. and Rubin, D.S., 1991, *Statistics for Management*, PHI, : New Delhi.

Feller, W., 1957, *An Introduction to Probability Theory and Its Applications*, John Wiley & Sons Inc. : New York.

Hooda, R.P. 2001, *Statistics for Business and Economics*. MacMillan India Limited, Delhi.

Gupta S.C., and V.K. Kapoor, 2005, *Fundamentals of Mathematical Statistics*, Sultan Chand & Sons, Delhi.

Gupta, S.P. *Statistical Methods*, 2000, Sultan Chand & Sons, Delhi.